

Review: Double Integrals

MATH 321 – Mathematical Statistics I

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October 17, 2026

Iterated Integrals

An **iterated integral** is evaluated from the inside out:

$$\int_a^b \int_c^d f(x, y) dy dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx.$$

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- The result of the inner integral is a function of x only, which we then integrate from a to b .

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The differentials are read from the inside out: the innermost differential (dy) goes with the innermost integral sign ($\int_c^d$).

Fubini's Theorem

Theorem (Fubini): If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.$$

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Over a rectangle, reversing the order of integration is easy: **the bounds just travel with their variable.**

Example 1: A Rectangular Region

Evaluate $\iint_R 6xy^2 dA$, where $R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 1\}$.

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Integrating y first:

$$\begin{aligned}\int_0^2 \int_0^1 6xy^2 dy dx &= \int_0^2 \left[2xy^3 \right]_{y=0}^{y=1} dx \\ &= \int_0^2 2x dx \\ &= \left[x^2 \right]_0^2 = 4.\end{aligned}$$

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Integrating x first:

$$\begin{aligned}\int_0^1 \int_0^2 6xy^2 dx dy &= \int_0^1 \left[3x^2y^2 \right]_{x=0}^{x=2} dy \\ &= \int_0^1 12y^2 dy \\ &= \left[4y^3 \right]_0^1 = 4.\checkmark\end{aligned}$$

A Shortcut for Products

If the region is a rectangle **and** the integrand factors as $f(x, y) = g(x) h(y)$, then

$$\int_a^b \int_c^d g(x) h(y) dy dx = \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right).$$

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For Example 1:

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Warning: This only works when *both* conditions hold. If the bounds depend on x or y , the integral does **not** split.

Rules for Setting Up Bounds

1. The **outer** bounds must be **constants**.
2. The **inner** bounds may depend on the **outer** variable, but never on the inner variable.
3. After evaluating the inner integral, the inner variable should be completely gone.
4. A double integral evaluates to a **number**, never a function of x or y .

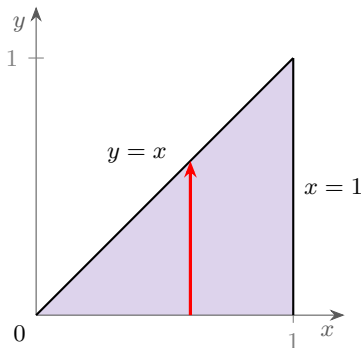
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Think in slices: with $dy\,dx$, fix x and draw a **vertical** line through the region. Its bottom and top give the y -bounds. With $dx\,dy$, fix y and draw a **horizontal** line. Its left and right ends give the x -bounds.

Example 2: A Triangular Region

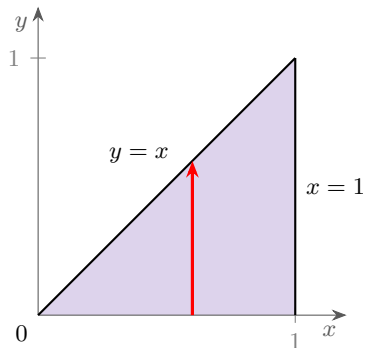
Let $f(x, y) = 6y$ for $A = \{(x, y) : 0 \leq y \leq x \leq 1\}$. Find $\iint_A f(x, y) dA$.



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Vertical slice at x : y runs from 0 up to x .



$$\begin{aligned}\int_0^1 \int_0^x 6y \, dy \, dx &= \int_0^1 \left[3y^2 \right]_{y=0}^{y=x} dx \\ &= \int_0^1 3x^2 \, dx \\ &= \left[x^3 \right]_0^1 = 1.\end{aligned}$$

Since $f \geq 0$ and it integrates to 1, f is a valid joint density.

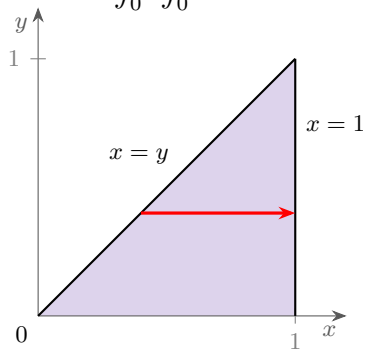
Reversing the Order of Integration

To switch from $dy\,dx$ to $dx\,dy$ (or vice versa):

1. **Read the region** off the current bounds and write it as a set of inequalities.
2. **Sketch the region.** Label each boundary curve and find where the curves intersect.
3. **Find the new outer bounds:** the smallest and largest values the new outer variable takes *anywhere* in the region. These must be constants.
4. **Find the new inner bounds:** fix the outer variable, draw a slice across the region, and see which curves it enters and exits through. Solve those curves for the inner variable.
5. **Check for a split:** if the entry or exit curve changes partway through, break the integral into pieces.

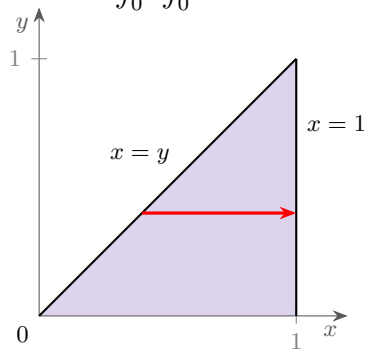
Example 2: Reversed

Rewrite $\int_0^1 \int_0^x 6y \, dy \, dx$ in the order $dx \, dy$.



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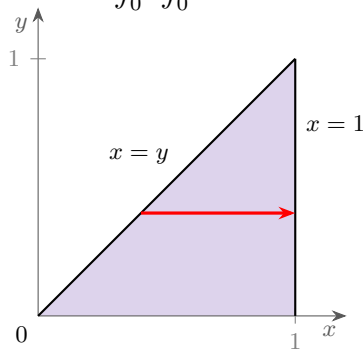
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- Horizontal slice at y : enter at $x = y$, exit at $x = 1$.

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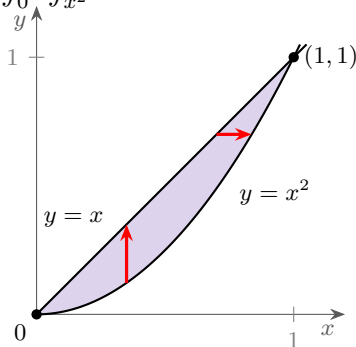
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$$\begin{aligned} \int_0^1 \int_y^1 6y \, dx \, dy &= \int_0^1 \left[6xy \right]_{x=y}^{x=1} dy \\ &= \int_0^1 (6y - 6y^2) dy \\ &= \left[3y^2 - 2y^3 \right]_0^1 = 1. \checkmark \end{aligned}$$

Example 3: Between Two Curves

Reverse the order of integration and evaluate both ways:

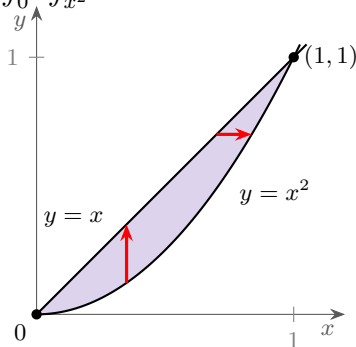
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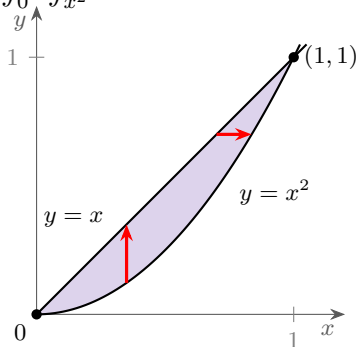


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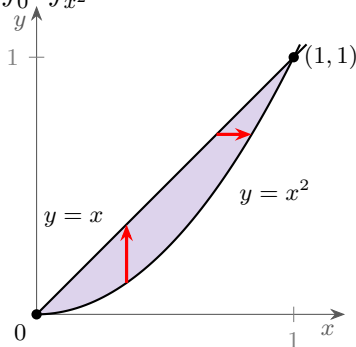
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$$\int_0^1 \int_{x^2}^x x \, dy \, dx = \int_0^1 \int_y^{\sqrt{y}} x \, dx \, dy$$

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Vertical slices ($dy\,dx$):

$$\int_0^1 \int_{x^2}^x x\,dy\,dx = \int_0^1 x(x - x^2)\,dx = \int_0^1 (x^2 - x^3)\,dx = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

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Horizontal slices ($dx\,dy$):

$$\begin{aligned} \int_0^1 \int_y^{\sqrt{y}} x\,dx\,dy &= \int_0^1 \left[\frac{x^2}{2} \right]_{x=y}^{x=\sqrt{y}} dy \\ &= \frac{1}{2} \int_0^1 (y - y^2)\,dy = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{1}{12}. \checkmark \end{aligned}$$

Why Reverse? Example 4

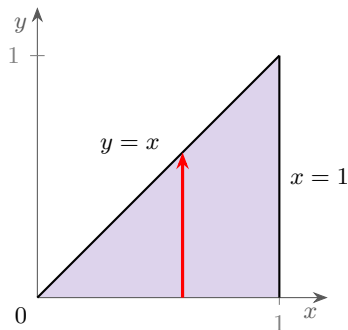
Evaluate $\int_0^1 \int_y^1 e^{x^2} dx dy$.

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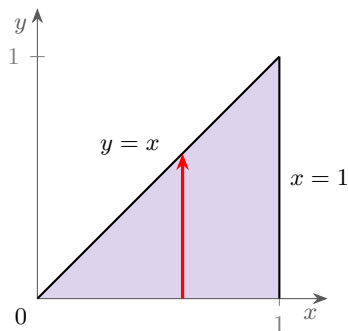
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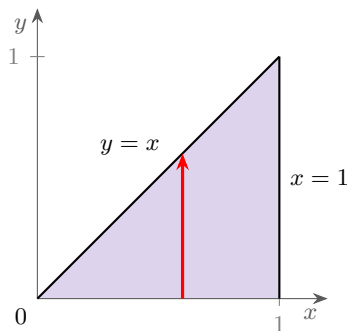
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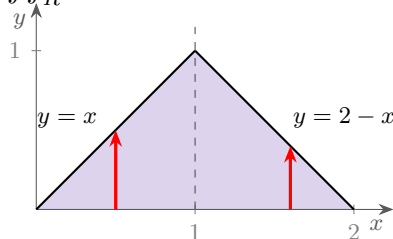
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$$\begin{aligned} \int_0^1 \int_0^x e^{x^2} dy dx &= \int_0^1 x e^{x^2} dx \\ &= \left[\frac{1}{2} e^{x^2} \right]_0^1 = \frac{e-1}{2}. \end{aligned}$$

Example 5: When One Order Needs a Split

Let R be the triangle with vertices $(0,0)$, $(2,0)$, and $(1,1)$. Evaluate

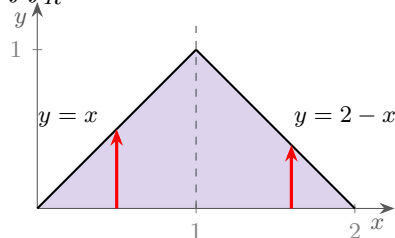
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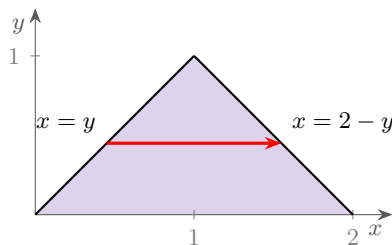
$$\iint_R y \, dA.$$



Vertical slices: the top curve changes at $x = 1$, so we must split:

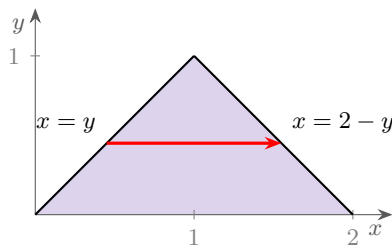
$$\begin{aligned} & \int_0^1 \int_0^x y \, dy \, dx + \int_1^2 \int_0^{2-x} y \, dy \, dx \\ &= \int_0^1 \frac{x^2}{2} \, dx + \int_1^2 \frac{(2-x)^2}{2} \, dx \\ &= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}. \end{aligned}$$

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Horizontal slices: for every $y \in [0, 1]$,
enter on $x = y$ and exit on $x = 2 - y$.
No split needed!

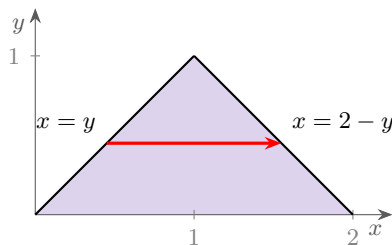
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Lesson: Before integrating, check whether one order gives a single integral. Choose that one.

Common Mistakes

- **Swapping bounds without re-deriving them.**

$\int_0^1 \int_0^x f \, dy \, dx \neq \int_0^x \int_0^1 f \, dx \, dy$. The right side isn't even a number!

- **Variable outer bounds.** If an outer bound contains x or y , the setup is wrong.
- **Mismatched differentials.** The order of dx and dy must match the bounds: $\int_0^1 \int_0^x 6y \, dx \, dy$ is wrong if you meant “ y from 0 to x .” The inner bounds belong to the inner differential.
- **Missing a split.** If a slice's entry or exit curve changes, you need more than one integral (or the other order).