

Lecture #9: Moment-Generating Functions

MATH 321 – Mathematical Statistics I

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Moment-Generating Functions

The **moment-generating function (MGF)** $M_X(t)$ of a random variable X is:

$$M_X(t) = E[e^{tX}],$$

provided this expectation is finite for all t in some open interval $(-h, h)$ around 0.

Moment-Generating Function Example

Example:

1. Find the moment-generating function for $X \sim \text{Exponential}(\beta)$.

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1. Find the moment-generating function for $X \sim \text{Exponential}(\beta)$.
2. Find the moment-generating function for $X \sim \text{Bernoulli}(p)$.
3. Find the moment-generating function for $X \sim \text{Binomial}(n, p)$.

Properties of Moment-Generating Functions

Theorem: If the moment-generating function of X exists, then:

1. $M_X(0) = 1$.
2. $\left. \frac{d}{dt} M_X(t) \right|_{t=0} = E[X]$.
3. $\left. \frac{d^2}{dt^2} M_X(t) \right|_{t=0} = E[X^2]$.
4. $\left. \frac{d^k}{dt^k} M_X(t) \right|_{t=0} = E[X^k]$, for any positive integer k .

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Example: Given that the MGF of $X \sim \text{Exponential}(\beta)$ is $M_X(t) = (1 - \beta t)^{-1}$ for $t < \frac{1}{\beta}$, use it to prove that $E[X] = \beta$ and $V(X) = \beta^2$.

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Theorem: Let X and Y be independent random variables. Let $Z = X + Y$. Then

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Corollary: Consider independent random variables $X_1, \dots, X_N$. Let $Z = \sum_{i=1}^N X_i$ be the sum of random variables. Then the MGF of Z is

$$M_Z(t) = \prod_{i=1}^N M_{X_i}(t).$$

What's the Deal with the MGF?

The moment-generating function of a random variable uniquely defines the distribution.

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Theorem: Let $M_X(t)$ and $M_Y(t)$ denote the moment-generating functions of random variables X and Y . If both moment-generating functions exist and $M_X(t) = M_Y(t)$ for all t in some open interval $(-h, h)$ around 0, then X and Y have the same probability distribution.

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The moment-generating function does not exist for all distributions. For example, the **Cauchy distribution** does not have an MGF.

Moment-Generating Functions of i.i.d. Random Variables

Theorem: Let $X_1, \dots, X_N$ be independent and identically distributed random variables with common moment-generating function $M_X(t)$, and let $Z = X_1 + \dots + X_N$. Then the moment-generating function of Z is

$$M_Z(t) = \left(M_X(t)\right)^N.$$

MGFs of Sums of Independent Random Variables

Examples:

1. Let $X_1, \dots, X_N$ be a sequence of i.i.d. Bernoulli random variables with parameter p . Let $Z = X_1 + \dots + X_N$. What is the distribution of Z ?

MGFs of Sums of Independent Random Variables

Examples:

1. Let $X_1, \dots, X_N$ be a sequence of i.i.d. Bernoulli random variables with parameter p . Let $Z = X_1 + \dots + X_N$. What is the distribution of Z ?
2. Let $X_1, \dots, X_N$ be a sequence of i.i.d. binomial random variables with parameters n and p . Let $Z = X_1 + \dots + X_N$. What is the distribution of Z ?

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3. Let $X_1, \dots, X_N$ be independent random variables with $X_i \sim \text{Poisson}(\lambda_i)$. Let $Z = X_1 + \dots + X_N$. What is the distribution of Z ?

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Examples:

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2. Let $X_1, \dots, X_N$ be a sequence of i.i.d. binomial random variables with parameters n and p . Let $Z = X_1 + \dots + X_N$. What is the distribution of Z ?
3. Let $X_1, \dots, X_N$ be independent random variables with $X_i \sim \text{Poisson}(\lambda_i)$. Let $Z = X_1 + \dots + X_N$. What is the distribution of Z ?
4. Let $X_1, \dots, X_N$ be independent random variables with $X_i \sim \text{Normal}(\mu_i, \sigma_i^2)$. Let $Z = X_1 + \dots + X_N$. What is the distribution of Z ?

The Characteristic Function

There is a related function that does exist for all distributions, the **characteristic function** $\varphi_X(t)$.

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The **characteristic function** $\varphi_X(t)$ of a random variable X is:

$$\varphi_X(t) = E[e^{itX}].$$

Properties of the Characteristic Function

Properties:

1. $\varphi_X(-it) = M_X(t)$, whenever the MGF exists.
2. $\frac{1}{i^k} \frac{d^k}{dt^k} \varphi_X(t) \Big|_{t=0} = E[X^k]$, for any positive integer k with $E[|X|^k] < \infty$.
3. Consider independent random variables $X_1, \dots, X_N$. Let $Z = \sum_{i=1}^N X_i$ be the sum of random variables. Then the characteristic function of Z is

$$\varphi_Z(t) = \prod_{i=1}^N \varphi_{X_i}(t).$$