

Lecture #8: Joint Distributions

MATH 321 – Mathematical Statistics I

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Multivariate Random Variables

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Multivariate Random Variables

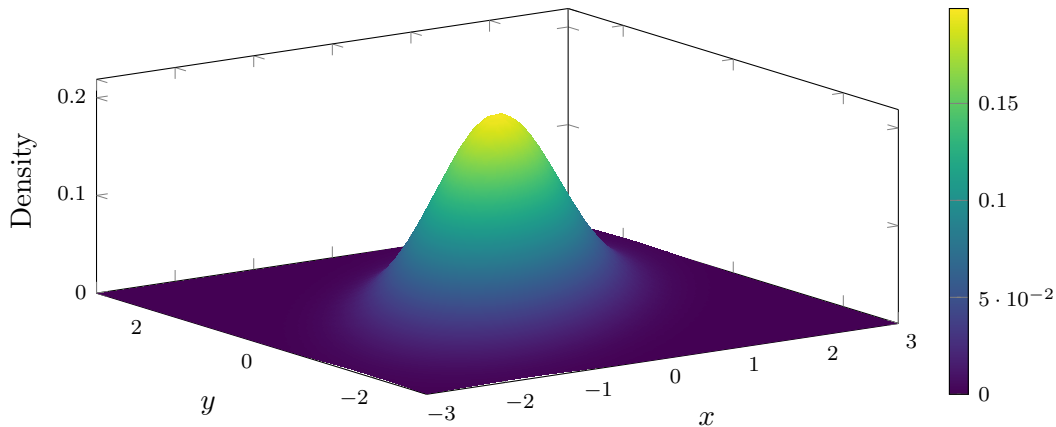
It's all well and good to consider random variables in isolation, but to do the actual interesting things you all want to do in probability and statistics, we need to understand how random variables relate to one another.

For instance, most of inferential statistics is built on the assumption of **independent and identically distributed (i.i.d.) random variables**.

Similarly, linear regression is built on a specific relationship of random variables:

$$Y = \beta_0 + \beta_1 X + \varepsilon, \quad \varepsilon \sim \text{Normal}(0, \sigma^2)$$

Bivariate Normal Distribution



Joint Probability Mass Functions

Let X and Y be two discrete random variables. The **joint probability mass function (PMF)** of X and Y is defined as:

$$\begin{aligned} p_{X,Y}(x,y) &= P[\{X = x\} \cap \{Y = y\}] \\ &= P[\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}]. \end{aligned}$$

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For any set $A \subseteq \mathbb{R}^2$,

$$P[(X,Y) \in A] = \sum_{(x,y) \in A} p_{X,Y}(x,y).$$

Note that these are valid PMFs! All properties apply!

Joint PMF Example

Example: Let X be a coin toss and Y be a die roll. Find $p_{X,Y}$.

		Y					
		1	2	3	4	5	6
X	H	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
	T	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

1. What is the probability of getting a heads and an even number?

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1. What is the probability of getting a heads and an even number?
2. Are X and Y independent?

Joint Probability Density Functions

Let X and Y be two continuous random variables. The **joint probability density function (PDF)** of X and Y is a function $f_{X,Y}(x,y)$ that can be integrated to yield a probability:

$$P[(X,Y) \in A] = \iint_A f_{X,Y}(x,y) dx dy,$$

for any event $A \subseteq \Omega_X \times \Omega_Y$.

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In particular, if $A = [a,b] \times [c,d]$, then

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Joint PDF Example

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$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{6}, & \text{for } 1 \leq x \leq 3, 1 \leq y \leq 4, \\ 0, & \text{otherwise.} \end{cases}$$

1. What is the probability that $X \geq 2$ and $Y \geq 2$?

Joint PDF Example

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1. What is the probability that $X \geq 2$ and $Y \geq 2$?
2. What is the probability that X and Y are less than 1 apart?

Marginal PMFs and Marginal PDFs

The **marginal probability mass function (PMF)** is defined as:

$$p_X(x) = \sum_{y \in \Omega_Y} p_{X,Y}(x, y) \quad \text{and} \quad p_Y(y) = \sum_{x \in \Omega_X} p_{X,Y}(x, y).$$

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The **marginal probability density function (PDF)** is defined as:

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Note that these are valid PMFs/PDFs! All properties apply!

Independent Random Variables

Two random variables X and Y are **independent** if their joint PMF/PDF factors into the product of the marginals:

$$p_{X,Y}(x,y) = p_X(x) p_Y(y) \quad \text{or} \quad f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

for all x and y .

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for all x and y .

Equivalently, $P[X \in A, Y \in B] = P[X \in A] P[Y \in B]$ for all events A and B .

Joint PMF Example

Example: Let X be a coin toss and Y be a die roll. Find $p_{X,Y}$.

		Y					
		1	2	3	4	5	6
X	H	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
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3. Find the marginal PMFs of X and Y .

Joint PDF Example

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3. Find the marginal PDFs of X and Y .

Extending to N Random Variables

A collection of random variables $X_1, X_2, \dots, X_N$ is called **independent and identically distributed (i.i.d.)** if:

1. All $X_1, X_2, \dots, X_N$ are mutually independent.

Extending to N Random Variables

A collection of random variables $X_1, X_2, \dots, X_N$ is called **independent and identically distributed (i.i.d.)** if:

1. All $X_1, X_2, \dots, X_N$ are mutually independent.
2. All $X_1, X_2, \dots, X_N$ have the same distribution, i.e. $F_{X_1}(x) = F_{X_2}(x) = \dots = F_{X_N}(x)$ for all x .

Joint Expectation

Let X and Y be two random variables. The **joint expectation** is

$$E[XY] = \sum_{y \in \Omega_Y} \sum_{x \in \Omega_X} xy p_{X,Y}(x, y),$$

if X and Y are discrete, or

$$E[XY] = \int_{\Omega_Y} \int_{\Omega_X} xy f_{X,Y}(x, y) dx dy,$$

if X and Y are continuous.

Joint Expectation Example

Example: Let $f_{X,Y}(x,y) = x + y$ for $0 \leq x, y \leq 1$ and 0 otherwise. Find $E[XY]$.

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$$\begin{aligned} E[XY] &= \int_0^1 \int_0^1 xy(x+y) dy dx = \int_0^1 \int_0^1 (x^2y + xy^2) dy dx \\ &= \int_0^1 \left(\frac{x^2}{2} + \frac{x}{3} \right) dx = \frac{1}{3}. \end{aligned}$$

Let X and Y be two random variables. Then the **covariance** of X and Y is

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)],$$

where $\mu_X = E[X]$ and $\mu_Y = E[Y]$.

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Note that $\text{Cov}(X, X) = V(X)$.

Theorem: Let X and Y be two random variables. Then

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

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If $X_1, X_2, \dots, X_n$ are random variables, then the matrix with (i, j) entry $\text{Cov}(X_i, X_j)$ is the **covariance matrix**, Σ .

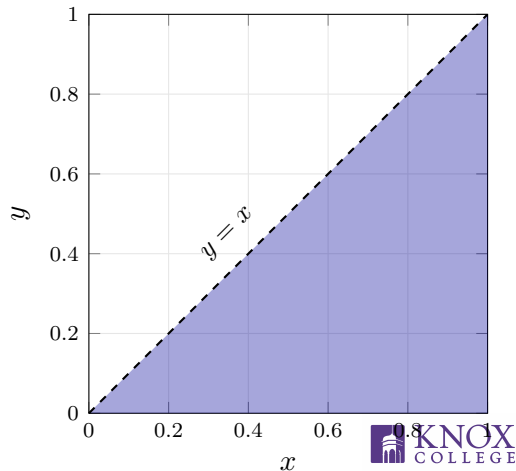
Covariance Example

Example: Let

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $Cov(X,Y)$.

1. Find $E[XY]$.



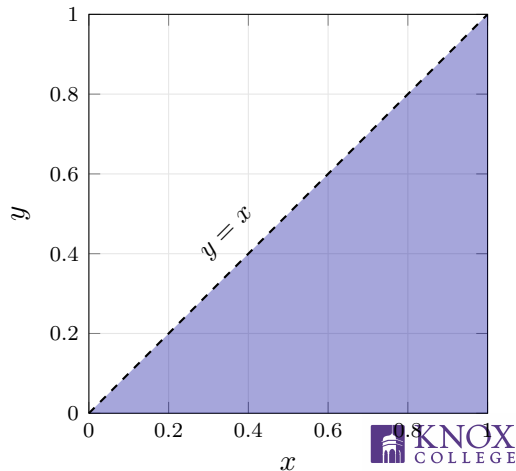
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2. Find the marginal PDFs and $E[X]$, $E[Y]$.



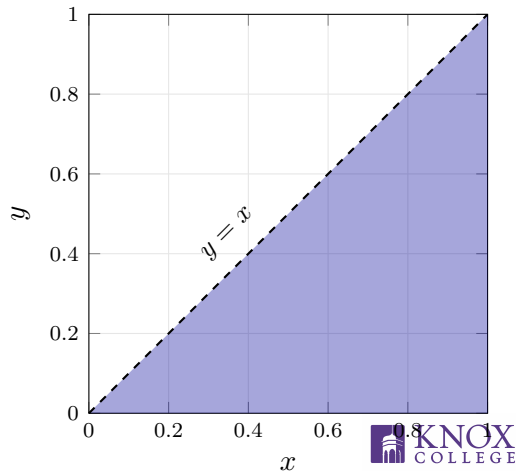
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Example: Let

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $Cov(X,Y)$.

1. Find $E[XY]$.
2. Find the marginal PDFs and $E[X]$, $E[Y]$.
3. Use $Cov(X,Y) = E[XY] - E[X]E[Y]$.



Joint Expectation and Variance

Theorem: For any X and Y ,

1. $E[X + Y] = E[X] + E[Y]$.
2. $V[X + Y] = V[X] + V[Y] + 2Cov(X, Y)$.

Joint Expectation and Variance

Theorem: Let $Y_1, Y_2, \dots, Y_n$ and $X_1, X_2, \dots, X_m$ be random variables with $E(Y_i) = \mu_i$ and $E(X_j) = \xi_j$. Define:

$$U_1 = \sum_{i=1}^n a_i Y_i \quad \text{and} \quad U_2 = \sum_{j=1}^m b_j X_j$$

for constants $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_m$.

Joint Expectation and Variance

Theorem: Then the following hold:

1. $E(U_1) = \sum_{i=1}^n a_i \mu_i.$
2. $V[U_1] = \sum_{i=1}^n a_i^2 V[Y_i] + 2 \sum_{1 \leq i < j \leq n} a_i a_j Cov(Y_i, Y_j).$
3. $Cov(U_1, U_2) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j Cov(Y_i, X_j).$

Theorem: If X and Y are independent, then

$$E[XY] = E[X]E[Y].$$

Covariance and Independence

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Question: Consider the following two statements:

- a. X and Y are independent.
 - b. $Cov(X, Y) = 0$.
1. Does statement (a) imply statement (b)?

Covariance and Independence

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Question: Consider the following two statements:

- a. X and Y are independent.
 - b. $Cov(X, Y) = 0$.
1. Does statement (a) imply statement (b)?
 2. Does statement (b) imply statement (a)?

Example of Zero Covariance but Non-Independent

Example: Let k be chosen uniformly at random from $\{1, 2, 3, 4\}$, and define X and Y by the table below.

	k			
	1	2	3	4
X	1	0	-1	0
Y	0	1	0	-1

1. What is the covariance of X and Y ?

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	k			
	1	2	3	4
X	1	0	-1	0
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1. What is the covariance of X and Y ?
2. Are X and Y independent?

The Correlation Coefficient

Let X and Y be two random variables. The **correlation coefficient** is

$$\rho = \frac{\text{Cov}(X, Y)}{\sqrt{V(X)V(Y)}}.$$

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Let X and Y be two random variables. The **correlation coefficient** is

$$\rho = \frac{\text{Cov}(X, Y)}{\sqrt{V(X)V(Y)}}.$$

Properties: (Assume $V(X), V(Y) > 0$.)

1. ρ is always between -1 and 1 , i.e. $-1 \leq \rho \leq 1$.
2. $\rho = 1$ if and only if $Y = aX + b$ for some $a > 0$ (e.g. $X = Y$).
3. $\rho = -1$ if and only if $Y = aX + b$ for some $a < 0$ (e.g. $X = -Y$).
4. If X and Y are independent, then $\rho = 0$.

Cauchy–Schwarz Inequality

Theorem: For any random variables X and Y ,

$$(E[XY])^2 \leq E[X^2]E[Y^2].$$

Conditional Probability

Recall that, if we have two events A and B and $P(B) \neq 0$, that the **conditional probability of A given B** is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Conditional Probability

Recall that, if we have two events A and B and $P(B) \neq 0$, that the **conditional probability of A given B** is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Can we generalize this to random variables, so that we can determine the distribution of X given $Y = y$?

Conditional PMFs

Let X and Y be two discrete random variables. The **conditional probability mass function (PMF) of X given Y** is

$$p_{X|Y}(x | y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}, \quad \text{provided } p_Y(y) > 0.$$

Conditional PMFs

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For fixed y , $p_{X|Y}(x | y)$ is a PMF.

Conditional PMFs

Theorem: Let X and Y be two discrete random variables. Let A be an event.

$$P[X \in A \mid Y = y] = \sum_{x \in A} p_{X|Y}(x \mid y) = \frac{\sum_{x \in A} p_{X,Y}(x, y)}{p_Y(y)},$$

and

$$\begin{aligned} P[X \in A] &= \sum_{x \in A} \sum_{y \in \Omega_Y} p_{X|Y}(x \mid y) p_Y(y) \\ &= \sum_{y \in \Omega_Y} P[X \in A \mid Y = y] p_Y(y). \end{aligned}$$

Conditional PMFs Example

		Y			
		1	2	3	4
X	1	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	0
	2	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
	3	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
	4	0	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$

1. Find the marginal PMF $p_X(x)$.

Conditional PMFs Example

		Y			
		1	2	3	4
X	1	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	0
	2	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
	3	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
	4	0	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$

1. Find the marginal PMF $p_X(x)$.
2. Find the conditional PMF $p_{X|Y}(x | 1)$.

Let X and Y be two continuous random variables. The **conditional probability density function (PDF) of X given Y** is

$$f_{X|Y}(x | y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \quad \text{provided } f_Y(y) > 0.$$

Let X and Y be continuous random variables. Then the **conditional cumulative distribution function of X given $Y = y$** is

$$F_{X|Y}(x | y) = \frac{\int_{-\infty}^x f_{X,Y}(t, y) dt}{f_Y(y)}.$$

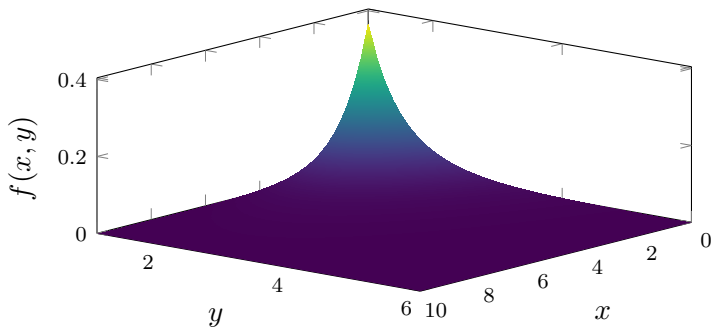
Theorem: Let X and Y be continuous random variables, and let A be an event.

i) $P(X \in A \mid Y = y) = \int_A f_{X|Y}(x \mid y) dx.$

ii) $P(X \in A) = \int_{\Omega_Y} P(X \in A \mid Y = y) f_Y(y) dy.$

Conditional CDF Example

Example: Suppose that $f(x, y) = \frac{e^{-\frac{x}{y}} e^{-y}}{y}$ for $0 < x, y < \infty$ and 0 otherwise. Find $P(X > 1 \mid Y = y)$.



Conditional Expectation

The **conditional expectation of X given $Y = y$** is

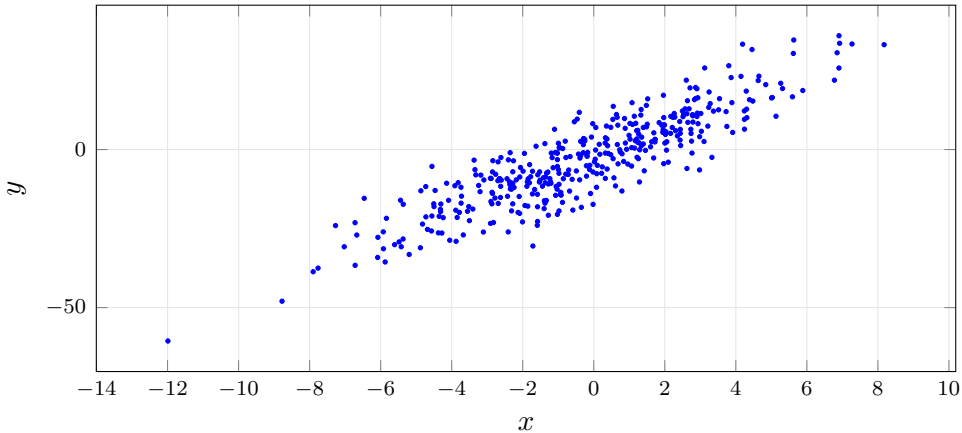
$$E[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$$

for discrete random variables, and

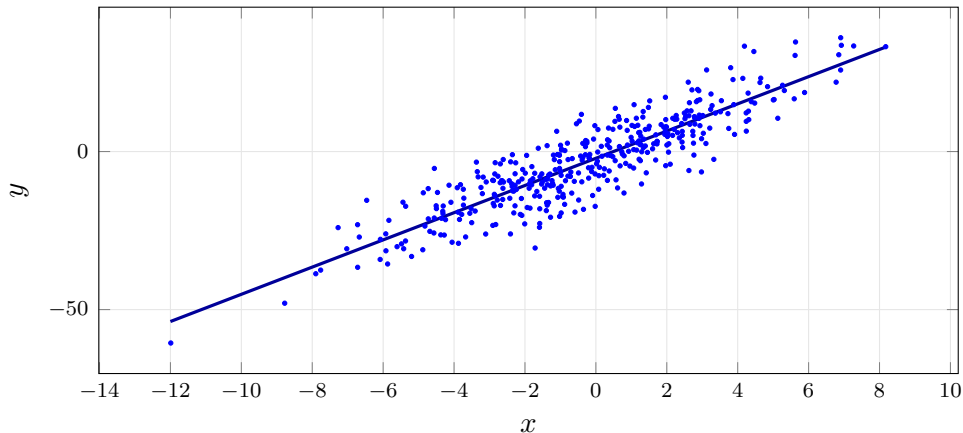
$$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) dx$$

for continuous random variables.

Conditional Expectation and Linear Regression



Conditional Expectation and Linear Regression



Conditional Expectation Example

		X				
		0.01	0.1	1	10	100
Y	10^4	0	0	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$
	10^2	$\frac{5}{12}$	$\frac{5}{18}$	$\frac{5}{36}$	0	0

1. Find $E[X \mid Y = 10^2]$.

Conditional Expectation Example

		X				
		0.01	0.1	1	10	100
Y	10^4	0	0	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$
	10^2	$\frac{5}{12}$	$\frac{5}{18}$	$\frac{5}{36}$	0	0

1. Find $E[X \mid Y = 10^2]$.
2. Find $E[X \mid Y = 10^4]$.

Law of Total Expectation

Theorem: The Law of Total Expectation states that

$$E[X] = \sum_y E[X \mid Y = y]p_Y(y),$$

or

$$E[X] = \int_{-\infty}^{\infty} E[X \mid Y = y]f_Y(y) dy.$$

Law of Total Expectation

Theorem: The Law of Total Expectation states that

$$E[X] = \sum_y E[X \mid Y = y]p_Y(y),$$

or

$$E[X] = \int_{-\infty}^{\infty} E[X \mid Y = y]f_Y(y) dy.$$

Corollary: Let X and Y be two random variables. Then

$$E[X] = E[E[X \mid Y]].$$

Law of Total Expectation Example

		X				
		0.01	0.1	1	10	100
Y	10^4	0	0	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$
	10^2	$\frac{5}{12}$	$\frac{5}{18}$	$\frac{5}{36}$	0	0

3. Find $E[X]$.