

Lecture #7: The Normal Distribution

MATH 321 – Mathematical Statistics I

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October 17, 2026



The Normal Distribution

A random variable Y is said to have a **normal (or Gaussian) probability distribution** if and only if, for $\sigma > 0$ and $-\infty < \mu < \infty$, the density function of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(y - \mu)^2}{2\sigma^2} \right\}, \quad -\infty < y < \infty.$$

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We write

$$Y \sim \text{Normal}(\mu, \sigma^2)$$

to say that Y is drawn from a normal distribution with parameters μ and σ^2 .

The Support of a Random Variable

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In the case of the binomial distribution, the support of Y is $\{0, 1, 2, \dots, n\}$.

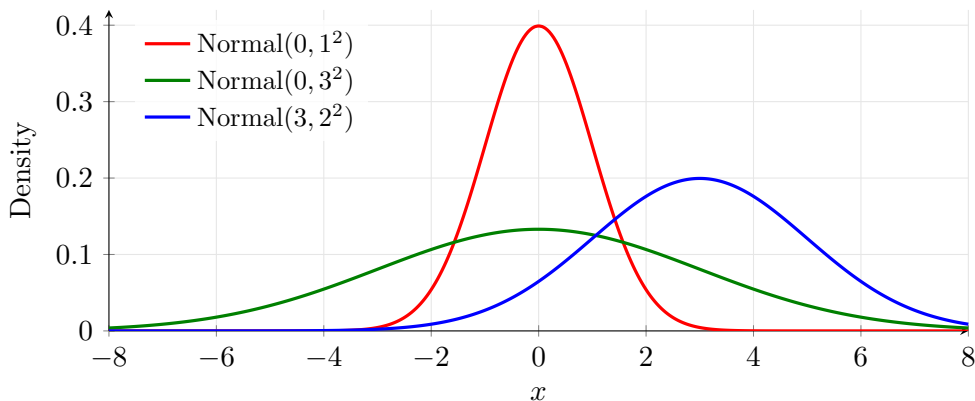
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In the case of the normal distribution, the support of Y is $\mathbb{R}$.

The Graph of the Normal Distribution



Facts about Definite Integrals

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If $f(x)$ is **even**, then for any $a > 0$:

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If $f(x)$ is **odd**, then for any $a > 0$:

$$\int_{-a}^a f(x) dx = 0$$

This also holds for $\int_{-\infty}^{\infty} f(x) dx$, provided $\int_{-\infty}^{\infty} |f(x)| dx < \infty$.

Expected Value and Variance of the Normal Distribution

Theorem: If $Y \sim \text{Normal}(\mu, \sigma^2)$, then

$$E[Y] = \mu \quad \text{and} \quad V[Y] = \sigma^2.$$

Proof of Expected Value of the Normal Distribution

Substitute $u = y - \mu$, so $du = dy$:

$$\begin{aligned} E[Y] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} y \exp \left\{ -\frac{(y - \mu)^2}{2\sigma^2} \right\} dy \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (u + \mu) \exp \left\{ -\frac{u^2}{2\sigma^2} \right\} du \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} u \exp \left\{ -\frac{u^2}{2\sigma^2} \right\} du \\ &\quad + \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \mu \exp \left\{ -\frac{u^2}{2\sigma^2} \right\} du \\ &= 0 + \mu \left(\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp \left\{ -\frac{u^2}{2\sigma^2} \right\} du \right) \\ &= \mu. \end{aligned}$$

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The first integral is 0 because the integrand is odd and $\int_{-\infty}^{\infty} |u| e^{-u^2/(2\sigma^2)} du < \infty$.

Proof of Variance of the Normal Distribution

Substitute $z = \frac{y - \mu}{\sigma}$, so $dy = \sigma dz$, then integrate by parts:

$$\begin{aligned} V[Y] &= \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^{\infty} (y - \mu)^2 \exp\left\{-\frac{(y - \mu)^2}{2\sigma^2}\right\} dy \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \left(-ze^{-\frac{z^2}{2}} \Big|_{-\infty}^{\infty} \right) + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \\ &= 0 + \sigma^2 \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \right) \\ &= \sigma^2. \end{aligned}$$

The Standard Normal Distribution

The **standard Normal (or Gaussian) random variable** Z has a PDF

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z^2}{2} \right\}.$$

That is, $Z \sim \text{Normal}(0, 1)$ is a Normal random variable with mean $\mu = 0$ and $\sigma^2 = 1$.

The Standard Normal Distribution

The cumulative distribution function (CDF) of the standard Normal is defined as the function $\Phi(\cdot)$

$$\Phi(z) \stackrel{\text{def}}{=} F_Z(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{t^2}{2}} dt.$$

CDF of an Arbitrary Normal Distribution

Theorem (CDF of an arbitrary Normal):

Let $Y \sim \text{Normal}(\mu, \sigma^2)$. Then

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Examples:

1. Find $P(Z \leq 2.00)$.
2. Let $Y \sim \text{Normal}(3, 4)$. Find $P(Y \leq 11)$.

Properties of Normal Distributions

Theorem: Let $Y \sim \text{Normal}(\mu, \sigma^2)$. Then the following results hold:

1. $\Phi(z) = 1 - \Phi(-z)$.
2. $P(Y \geq b) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right)$.
3. $P(|Y| \geq b) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right) + \Phi\left(\frac{-b-\mu}{\sigma}\right)$, for $b \geq 0$.

Normal Distribution Example

Intelligence Quotient (IQ) scores are defined to follow a Normal distribution with a mean of 100 and a standard deviation of 15. Use this information to answer the following questions:

1. Find the probability that a randomly chosen person has an IQ less than 92.

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1. Find the probability that a randomly chosen person has an IQ less than 92.
2. Find the probability that a randomly chosen person has an IQ of 120 or greater.
3. Find the probability that a randomly chosen person has an IQ between 83 and 114.

Skewness and Kurtosis

For a random variable X with PDF $f_X(x)$, define the following moments as:

$$\text{mean: } E[X] \stackrel{\text{def}}{=} \mu,$$

$$\text{variance: } E[(X - \mu)^2] \stackrel{\text{def}}{=} \sigma^2,$$

$$\text{skewness: } E\left[\left(\frac{X - \mu}{\sigma}\right)^3\right] \stackrel{\text{def}}{=} \gamma,$$

$$\text{kurtosis: } E\left[\left(\frac{X - \mu}{\sigma}\right)^4\right] \stackrel{\text{def}}{=} \kappa.$$

Skewness and Kurtosis

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The normal distribution has skewness 0 and excess kurtosis 0.

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4. What IQ score is greater than 40% of people?
5. What IQ score is less than the top 15% of people?
6. What IQ scores are the bounds of the middle 95% of people?

Percentiles of Probability Distributions

Let X be a continuous random variable with probability density function f_X . The **median of X** is a point $c \in \mathbb{R}$ such that

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Theorem: The **median** of a random variable X is the point c such that

$$F_X(c) = \frac{1}{2}.$$

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The k^{th} **percentile** of a random variable X is the point c such that

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$$F_X(c) = \frac{k}{100}.$$

The **quantile function** is the inverse of the CDF (when F_X is strictly increasing). In general, it is defined as $F_X^{-1}(p) = \inf\{x : F_X(x) \geq p\}$. These are hard to compute for most distributions, so we tend to use R.

Percentile Example

A train arrives at the station between 0 and 15 minutes late, distributed uniformly:

1. What is the median number of minutes the train is late?

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2. What is the 35th percentile of the amount of time the train will be late?

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Intelligence Quotient (IQ) scores are defined to follow a Normal distribution with a mean of 100 and a standard deviation of 15. Use this information to answer the following questions:

7. Mensa International famously only allows people who score at the 98th percentile or higher on a standardized, supervised IQ test to join. What IQ score must you have to join Mensa?