

Lecture #6: Continuous Probability Distributions

MATH 321 – Mathematical Statistics I

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October 17, 2026



Recall: Axioms of Probability

Definition: A **probability space** is a triple $(\Omega, \mathcal{F}, P)$, where Ω is the **sample space**, $\mathcal{F}$ is the **event space**, and P is a **probability law (or measure)**.

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Definition: A **probability law** is a function $P : \mathcal{F} \rightarrow [0, 1]$ that maps an event A to a real number in $[0, 1]$. The function must satisfy the **axioms of probability**:

1. **Non-Negativity:** $P(A) \geq 0$, for any $A \in \mathcal{F}$.
2. **Normalization:** $P(\Omega) = 1$.
3. **Additivity:** For any pairwise disjoint events $\{A_1, A_2, A_3, \dots\}$, it must be true that

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

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Let's now generalize this to continuous random variables.

Probability Density Functions

A **probability density function** f_X of a random variable X is a mapping $f_X : X(\Omega) \rightarrow \mathbb{R}$, with the properties

1. **Non-Negativity:** $f_X(x) \geq 0$ for all $x \in X(\Omega)$.
2. **Unity:** $\int_{X(\Omega)} f_X(x) dx = 1$.
3. **Measure of a Set:** $P(X \in A) = \int_A f_X(x) dx$.

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2. **Unity:** $\int_{X(\Omega)} f_X(x) dx = 1$.
3. **Measure of a Set:** $P(X \in A) = \int_A f_X(x) dx$.

We take $f_X(x) = 0$ for any $x \notin X(\Omega)$.

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Let X be a continuous random variable. The **probability density function of X** is a function $f_X : X(\Omega) \rightarrow \mathbb{R}$ that, when integrated over an interval $[a, b]$, yields the probability of obtaining $a \leq X \leq b$:

$$P[a \leq X \leq b] = \int_a^b f_X(x) dx.$$

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The probability density function uniquely determines the *distribution* of a random variable. If $X(\Omega) = Y(\Omega)$ and $f_X(t) = f_Y(t)$ for all $t \in X(\Omega)$, then X and Y have the same distribution (but need not be equal as random variables).

Example: Probability Density Function

Example: Let $f_X(x) = 3x^2$ with $X(\Omega) = [0, 1]$.

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3. What does this mean graphically?
4. What is $P(0 \leq X < 0.5)$?

Cumulative Distribution Function

Let X be a continuous random variable. The **cumulative distribution function (CDF) of X** is:

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The **survival function of X** is:

$$S_X(x) = 1 - F_X(x) \stackrel{\text{def}}{=} P[X > x] = \int_x^{\infty} f_X(t) dt.$$

The Cumulative Distribution Function vs. Probability Density Function

The **probability density function (PDF)** is the derivative of the cumulative distribution function (CDF):

$$f_X(x) = \frac{d}{dx} F_X(x),$$

provided F_X is differentiable at x .

If F_X has a jump discontinuity at $x = x_0$, then $f_X(x)$ contains the term

$$P[X = x_0] \delta(x - x_0).$$

Example: Cumulative Distribution Function

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Properties of the Cumulative Distribution Function

Theorem: Let X be a random variable (either continuous or discrete), then the CDF of X has the following properties:

1. The CDF is **nondecreasing**.
2. The CDF approaches 1 as x approaches ∞ : $\lim_{x \rightarrow \infty} F_X(x) = 1$.
3. The CDF approaches 0 as x approaches $-\infty$: $\lim_{x \rightarrow -\infty} F_X(x) = 0$.

Continuity of the Cumulative Distribution Function

Theorem: For any random variable X (discrete or continuous), $F_X(x)$ is always **right-continuous**. That is,

$$F_X(b) = F_X(b^+) \stackrel{\text{def}}{=} \lim_{h \rightarrow 0^+} F_X(b + h).$$

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Theorem: For any random variable X (discrete or continuous), $P(X = b)$ is:

$$P[X = b] = \begin{cases} F_X(b) - F_X(b^-), & \text{if } F_X \text{ is discontinuous at } x = b, \\ 0, & \text{otherwise,} \end{cases}$$

where $F_X(b^-) \stackrel{\text{def}}{=} \lim_{h \rightarrow 0^+} F_X(b - h)$.

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$$E[X] = \int_{X(\Omega)} x f_X(x) dx.$$

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The **expectation** of a continuous random variable X is:

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Theorem: Let $g : X(\Omega) \rightarrow \mathbb{R}$ be a function and X be a continuous random variable. Then:

$$E[g(X)] = \int_{X(\Omega)} g(x) f_X(x) dx.$$

The n^{th} Moment of a Continuous Random Variable

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The n^{th} **central moment** of a continuous random variable X is

$$E[(X - \mu)^n] = \int_{X(\Omega)} (x - \mu)^n f_X(x) dx,$$

where $\mu = E[X]$.

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$$V[X] = \int_{X(\Omega)} (x - \mu)^2 f_X(x) dx.$$

The formulas concerning the variance and the expected value hold, such as $V(X) = E(X^2) - [E(X)]^2$.

Example: Expectation and Variance

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11. What is the variance of X , $V(X)$?

The Uniform Distribution

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The probability density function of X is:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise,} \end{cases}$$

where $[a, b]$ is the interval on which X is defined.

The Uniform Distribution

We write

$$X \sim \text{Uniform}(a, b)$$

to mean that X is drawn from a uniform distribution on an interval $[a, b]$.

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The variance of a uniformly distributed random variable is:

$$V(X) = \frac{(b - a)^2}{12}.$$

Example: The Uniform Distribution

Example: Suppose a train has a uniform probability of arriving between 0 and 10 minutes late.

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The Exponential Distribution

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The probability density function of X is:

$$f_X(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}}, & x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\beta > 0$ is a parameter.

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The Gamma Distribution

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The probability density function of X is:

$$f_X(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, & x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ (shape) and $\beta > 0$ (scale) are two parameters.

The Gamma Function

$\Gamma(z)$ is the **gamma function**, defined for $z > 0$ as:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt.$$

More usefully, for $z > 1$, this function has the property:

$$\Gamma(z) = (z-1)\Gamma(z-1).$$

If $z \in \mathbb{Z}^+$, then $\Gamma(z) = (z-1)!$. Also, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

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The Beta Distribution

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The probability density function of X is:

$$f_X(x) = \begin{cases} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, & 0 \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$ are two parameters.

The Beta Distribution

We write

$$X \sim \text{Beta}(\alpha, \beta)$$

to mean that X is drawn from a Beta distribution with parameters α and β .

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$$V(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$